Available online at www.sciencedirect.com
ScienceDirect
Journal of Geometry and Physics 57 (2006) 101–119
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

On quasi-triviality and integrability of a class of scalar evolutionary PDEs

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Received 4 February 2006; accepted 9 February 2006
Available online 20 March 2006

Abstract

For a certain class of perturbations of the equation $u_t = f(u)u_x$, we prove the existence of change of coordinates, called quasi-Miura transformations, that reduce these perturbed equations to the unperturbed one. As an application, we propose a criterion for the integrability of these equations.

MSC: primary 37K10; secondary 35Q53

Keywords: Quasi-triviality; Scalar evolutionary PDEs; Integrability

1. Introduction

The notion of quasi-triviality for a certain class of evolutionary partial differential equations (PDEs) was introduced in [8]. Let us first explain this notation by looking at the example of the Kortweg-de Vries (KdV) equation

$$u_t = uu_x + \frac{\epsilon^2}{12}u_{xxx}, \tag{1.1}$$

where $u = u(x,t)$ and ϵ is the dispersion parameter. By the definition of [8], this equation is quasi-trivial, which means that we can perform a change of the dependent variable

$$v = u + \frac{\epsilon^2}{24}\partial_x^2\left[-\log u_x + \frac{\epsilon^2}{240}\left(5\frac{u^{(4)}}{u_x^2} - 9\frac{u_{xx}u_{xxx}}{u_x^3} + 4\frac{u_{xx}^3}{u_x^4}\right) + O(\epsilon^4)\right] \tag{1.2}$$

such that, in the new dependent variable, the KdV equation is formally reduced to the dispersionless equation

$$v_t = vv_x. \tag{1.3}$$

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0393-0440/$ - see front matter
doi:10.1016/j.geomphys.2006.02.005

Here, the transformation (1.2) is understood to be a formal power series of ϵ. The inverse transformation has the form

$$u = v + \frac{\epsilon^2}{24}\partial_x^2\left[\log v_x + \frac{\epsilon^2}{240}\left(5\,\frac{v^{(4)}}{v_x^2} - 21\,\frac{v_{xx}v_{xxx}}{v_x^3} + 16\,\frac{v_{xx}^3}{v_x^4}\right) + O(\epsilon^4)\right]. \tag{1.4}$$

This property of the KdV equation was first observed in [1] (see also [8]). It corresponds to the genus expansion formula for the free energy of the two-dimensional (2D) topological gravity [5,10,14] which provides one of the important links between 2D topological field theory and integrable hierarchies [7,8,10,15,21]. In [8,9] it was proved that such a property of quasi-triviality is also shared by a wide class of bihamiltonian integrable hierarchies, and it plays an important role in the study of the problem of the classification of these integrable hierarchies. In a very recent paper [6], Dubrovin showed that, at the approximation up to ϵ^4, any Hamiltonian perturbation of the Eq. (1.3) is quasi-trivial.

In the present paper we consider, without the assumption of possession of Hamiltonian structures, the property of quasi-triviality for a class of generalized scalar evolutionary PDEs of the form

$$\begin{aligned} u_t &= f(u)u_x + \epsilon(f_1(u)u_{xx} + f_2(u)u_x^2) \\ &\quad + \epsilon^2(f_3(u)u_{xxx} + f_4(u)u_xu_{xx} + f_5(u)u_x^3) + \cdots, \qquad f'(u) \neq 0. \end{aligned} \tag{1.5}$$

Here, the right-hand side of the equation is a power series of the parameter ϵ, the coefficients of ϵ^k are graded homogeneous polynomials of degree $k+1$ of the variables $u_x, u_{xx}, \ldots$ with $\deg \partial_x^k u = k$, and the coefficients of these polynomials are assumed to be smooth functions of u. Note that, when the power series of ϵ is a polynomial, (1.5) becomes a usual evolutionary PDE.

We are going to prove the quasi-triviality of the Eq. (1.5) according to the following definition of [8]:

Definition 1.1. The generalized evolutionary PDE (1.5) is called quasi-trivial if there exists a quasi-Miura transformation of the form

$$u = v + \sum_{k\geq 1}\epsilon^k F_k(v, v_x, \ldots, \partial_x^{m_k}v) \tag{1.6}$$

that formally reduces it to the equation

$$v_t = f(v)v_x. \tag{1.7}$$

Here, $F_k,\ k \geq 1$ are smooth functions and m_k are some positive integers.

We call a quasi-Miura transformation of the form (1.6) that transforms the Eq. (1.5) to its leading term Eq. (1.7) the *reducing transformation* of (1.5). The transformation (1.6) has an inverse of the same form. Note that, in the original definition of quasi-triviality given in [8,9], the coefficients F_k of the quasi-Miura transformation are required to be rational in the variables $u_x, u_{xx}, \ldots$. Here we slightly generalize this definition, see Section 4 for the explicit description of the quasi-Miura transformations that we will encounter in this paper.

The main motivation of this work originates from our attempt to generalize the classification scheme given in [8] for a class of bihamiltonian evolutionary PDEs; we expect that the requirement of bihamiltonian property can be replaced by a weaker one. The first step along this line is to find a more general class of evolutionary PDEs that possess the quasi-triviality property, since this property of the equations plays an important role in the classification scheme [8,9,16]. Another motivation for our study comes from the application of the reducing transformation to the perturbative study of solutions of the Eq. (1.5); see for example [6], in which such reducing transformations are used to study the critical behavior of solutions of the perturbed equations.

A direct consequence of the quasi-triviality property is the existence of infinitely many flows of the form

$$u_s = h(u)u_x + \sum_{k\geq 2}\epsilon^{k-1}W_k(u, u_x, \ldots, \partial^{n_k}u) \tag{1.8}$$

that commute with the flow (1.5), where $h(u)$ is an arbitrary smooth function. By imposing the conditions of polynomial dependence of the functions W_k on the variables $u_x, \ldots, \partial^{n_k}u$, we propose a criterion for the formal integrability of the Eq. (1.5).

The plan of the paper is as follows. In Sections 2 and 3, we introduce some basic notations, including the Definition 3.9 of formal integrability for the equations of the form (1.5), and prove an important property of the formally integrable equations in Theorem 3.10. In Section 4, we prove the quasi-triviality of the equations of the form (1.5) and summarize the main results in Theorems 4.3 and 4.5. In Section 5, we describe a criterion of formal integrability and illustrate this notation by some examples. Finally, in the conclusion we discuss the generalization of the quasi-triviality property to certain systems of evolutionary PDEs.

2. Miura-type transformations

We first introduce some notations that will be used in this paper; see [8] for more detailed expositions. Let $u(x)$ be a smooth function of a real variable x and denote $u_0 = u(x),\ u_s = \partial_x^s u(x),\ s \geq 1$. We define the ring $\mathscr{R}$ of differential polynomials of $u(x)$ as

$$\mathscr{R} = C^\infty(u_0)[u_1, u_2, \ldots]. \tag{2.1}$$

It is a graded ring with $\deg u_i = i,\ i \geq 1$, and $\deg h(u_0) = 0$ for any smooth function h. We denote by $\mathscr{A}$ the ring of formal power series of an indeterminate ϵ of the form

$$f = \sum_{i\geq 0} f_i(u_0, \ldots, u_i)\epsilon^i \tag{2.2}$$

where $f_i \in \mathscr{R}$ are homogeneous differential polynomials of degree i.

The derivations of $\mathscr{A}$ form a Lie algebra

$$\mathfrak{g} = \left\{ \hat{X} = \sum_{s\geq 0} X_s \frac{\partial}{\partial u_s} \,\middle|\, X_s \in \mathscr{A} \right\} \tag{2.3}$$

with Lie bracket

$$\left[\sum_{s\geq 0} X_s \frac{\partial}{\partial u_s}, \sum_{s\geq 0} Y_s \frac{\partial}{\partial u_s} \right] = \sum_{s\geq 0} Z_s \frac{\partial}{\partial u_s}, \quad \text{where } Z_s = \sum_{t\geq 0} \left(X_t \frac{\partial Y_s}{\partial u_t} - Y_t \frac{\partial X_s}{\partial u_t} \right). \tag{2.4}$$

We regard the ring $\mathscr{A}$ as the coordinate ring of an infinite dimensional manifold, and the Lie algebra $\mathfrak{g}$ as the Lie algebra of vector fields of this manifold.

Introduce the differential operator $\partial_x \in \mathfrak{g}$ by

$$\partial_x = \sum_{s\geq 0} u_{s+1} \frac{\partial}{\partial u_s}. \tag{2.5}$$

An element $\hat{X} \in \mathfrak{g}$ is called an evolutionary vector field if $[\hat{X}, \partial_x] = 0$, which implies that

$$\hat{X} = \sum_{s\geq 0} \left(\partial_x^s X\right) \frac{\partial}{\partial u_s} \quad \text{for certain } X \in \mathscr{A}. \tag{2.6}$$

The function X is called the component of the evolutionary vector field $\hat{X}$. Denote

$$\mathscr{E} = \{\text{the vector space of evolutionary vector fields}\}. \tag{2.7}$$

Then we readily have the following proposition:

Proposition 2.1. *The vector space $\mathscr{E}$ is a Lie subalgebra of $\mathfrak{g}$ with the center $\{a\,\partial_x | a \in \mathbb{R}\}$, and its Lie bracket can be expressed as*

$$[\hat{X}, \hat{Y}] = \hat{Z}, \quad \text{where } Z = \hat{X}(Y) - \hat{Y}(X),\ \forall\, \hat{X}, \hat{Y} \in \mathscr{E}. \tag{2.8}$$

Since the map $X \mapsto \hat{X}$ is a bijection between $\mathscr{A}$ and $\mathscr{E}$, we can pull back the Lie bracket of $\mathscr{E}$ to $\mathscr{A}$. Then we obtain a Lie bracket on $\mathscr{A}$

$$[X, Y] = \hat{X}(Y) - \hat{Y}(X) = \sum_{s\geq 0} \left(\left(\partial_x^s X\right) \frac{\partial Y}{\partial u_s} - \left(\partial_x^s Y\right) \frac{\partial X}{\partial u_s} \right). \tag{2.9}$$

Henceforth, we will also call an element X of $\mathscr{A}$ a vector field.

The subalgebra $\mathscr{B}$ of the Lie algebra $(\mathscr{A}, [\cdot, \cdot])$ defined by

$$\mathscr{B} = \left\{ X = \sum_{i\geq 1} f_i(u_0, \ldots, u_i)\epsilon^i \in \mathscr{A} \,|\, \deg f_i = i \right\} \tag{2.10}$$

corresponds to the class of evolutionary PDEs that we will study in this paper. Namely, if we express X in the form

$$X = \epsilon f(u)u_x + \epsilon^2 (f_1(u)u_{xx} + f_2(u)u_x^2) + \epsilon^3 (f_3(u)u_{xxx} + f_4(u)u_x u_{xx} + f_5(u)u_x^3) + \cdots, \tag{2.11}$$

and assume that it satisfies the condition $f'(u) \neq 0$, then

$$u_t = \epsilon^{-1} X \tag{2.12}$$

is just Eq. (1.5). A vector field $X \in \mathscr{B}$ that satisfies the above condition will be called generic. We call $\epsilon f(u)u_x$ the leading term of the vector field X, and Eq. (1.7) the leading term equation of (2.12).

In the next section, we will consider the normal forms of the generalized evolutionary PDEs of the form (1.5) under the Miura-type transformations.

Definition 2.2 (*[8]*). A Miura-type transformation is a transformation on the ring $\mathscr{A}$ which has the form

$$u \mapsto \tilde{u} = X_0(u) + \epsilon X_1(u)u_1 + \epsilon^2 \left(X_2(u)u_2 + X_3(u)u_1^2 \right) + \cdots, \qquad \frac{\partial X_0(u)}{\partial u} \neq 0. \tag{2.13}$$

A Miura-type transformation is an automorphism of the ring $\mathscr{A}$, and it induces an automorphism of the Lie algebra $\mathscr{B}$. It is easy to see that all such transformations form a group which is called the Miura group [8]. It is the semi-direct product of two subgroups. The first subgroup is the local diffeomorphism group of $\mathbb{R}$

$$u \mapsto \tilde{u} = f(u), \quad \text{where } \frac{\partial f}{\partial u} \neq 0. \tag{2.14}$$

The second subgroup is formed by the Miura-type transformations with $X_0(u) = u$. One can prove that any Miura-type transformation with $X_0(u) = u$ can be expressed as

$$g: \; u \mapsto \tilde{u} = e^{\hat{Y}} u = u + \hat{Y}(u) + \frac{1}{2}\hat{Y}(\hat{Y}(u)) + \frac{1}{6}\hat{Y}(\hat{Y}(\hat{Y}(u))) + \cdots \tag{2.15}$$

where $Y = \epsilon Y_1(u)u_1 + \epsilon^2 \left(Y_2(u)u_2 + Y_3(u)u_1^2\right) + \cdots \in \mathscr{B}$. The automorphism of $\mathscr{B}$ induced by this Miura-type transformation g is given by

$$X \mapsto g(X) = e^{-\mathrm{ad}_Y} X = X - [Y, X] + \frac{1}{2}[Y, [Y, X]] - \frac{1}{6}[Y, [Y, [Y, X]]] + \cdots. \tag{2.16}$$

Here, the vector field $g(X)$ is obtained by first expressing the vector field X in the new coordinate $\tilde{u}$, then re-denoting $\tilde{u}$ by u. We call the transformations of these two subgroups the Miura-type transformations of the first and second kind, respectively.

3. Formal symmetries and integrability

In this section, we first give the definition of formal symmetries for a generalized evolutionary PDE of the form (1.5) (or, equivalently, (2.12) for a generic vector field $X \in \mathscr{B}$) then, based on the properties of the formal symmetries, we introduce the notion of formal integrability. We show that the property of formal integrability of a equation of the form (1.5) is equivalent to the existence of a unique reduced form of the equation under Miura-type transformations.

Definition 3.1. Given a generic vector field $X \in \mathscr{B}$ of the form (2.11), a formal symmetry of Eq. (1.5) is a flow of the form

$$\epsilon \partial_s u = Y, \quad Y \in \mathscr{B} \tag{3.1}$$

which commutes with (1.5). We will also call Y a formal symmetry of the vector field X.

This definition is adopted from the usual definition of generalized infinitesimal symmetries of an evolutionary PDE. It differs from the usual one at the following two points. Firstly, even when (1.5) is a usual evolutionary PDE whose right-hand side is truncated, i.e. a polynomial of the finite number of variables $u_1, \ldots, u_m$ for certain positive integer m, its formal symmetries are not necessarily truncated. In the usual definition, an infinitesimal symmetry depends only on a finite number of the variables $u_i,\ i \geq 1$. Secondly, the right-hand side of a formal symmetry (3.1) is required to have the same form (2.11) as the vector field X (except for the condition $f'(u) \neq 0$). These features of the formal symmetries will play a crucial role in our discussion of formal integrability of the equations of the form (1.5).

Before the exposition of properties of the formal symmetries, let us first do some preparations. Let $\mathscr{P}$ be the set of all ordered sequences of integers $\lambda = (\lambda_1, \lambda_2, \ldots)$ with the properties

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq 0, \quad |\lambda| = \sum_{i \geq 1} \lambda_i < \infty. \tag{3.2}$$

We call $|\lambda|$ the degree of λ, and denote by $\mathscr{P}^d$ the subset of degree d elements of $\mathscr{P}$. Each $\lambda \in \mathscr{P}$ is associated with a unique monomial

$$u_\lambda = \prod_{i \geq 1} u_{\lambda_i} \in \mathscr{R}. \tag{3.3}$$

Given two elements $\lambda, \mu \in \mathscr{P}^d$, we say $\lambda > \mu$ (resp. $\lambda < \mu$) if the first non-zero entry of the sequence $(\lambda_1 - \mu_1, \lambda_2 - \mu_2, \lambda_3 - \mu_3, \ldots)$ is greater (resp. less) than 0. By using this ordering of $\mathscr{P}^d$, we can define in a natural way the highest order term of a homogeneous differential polynomial $X \in \mathscr{R}$ of degree d. For example, the highest order term of

$$X = X_1(u)u_4u_1 + X_2(u)u_3u_2 + X_3(u)u_2u_2u_1 \tag{3.4}$$

is $X_1(u)u_4u_1$, and we denote the sum of all other terms in X by *l.o.t.*

Lemma 3.2. *Let $X \in \mathscr{R}$ be homogeneous of degree d of the form*

$$X = f(u) \prod_{k=1}^{m} (u_k)^{\alpha_k} + l.o.t, \tag{3.5}$$

then we have

$$[uu_1, X] = \left(\sum_{k=2}^{m} \alpha_k + d - 1 \right) f(u) u_1 \prod_{k=1}^{m} (u_k)^{\alpha_k} + l.o.t. \tag{3.6}$$

Proof. After the substitution of (3.5) into (2.9), we can prove the lemma by a straightforward computation. □

Lemma 3.3. *Assume that $X \in \mathscr{R}$ is given as in the above lemma. Then $[uu_1, X] = 0$ if and only if $d = 1$, i.e. X has the form $X = f(u)u_1$.*

Proof. Let X have the form (3.5) and satisfy $[uu_1, X] = 0$, then from Lemma 3.2 we obtain $\alpha_1 = 1, \alpha_2 = \cdots = \alpha_m = 0$, so $X = f(u)u_1$. Conversely, it is obvious that the identity $[uu_1, X] = 0$ holds true for any vector field X of the form $f(u)u_1$. The lemma is proved. □

Lemma 3.4. *For a homogeneous differential polynomial $Y \in \mathscr{R}$ of degree $d > 2$, if the highest order term of Y has a factor u_1, then we can find a homogeneous differential polynomial $X \in \mathscr{R}$ of degree $d - 1$ such that $[u_0 u_1, X]$ has the same highest order term as that of Y.*

Proof. When $d \geq 2$, we always have $\left(\sum_{k=2}^{m} \alpha_k + d - 1\right) \neq 0$. So, the lemma immediately follows from Lemma 3.2. □

Theorem 3.5. *Let $Y \in \mathscr{B}$ be a symmetry of a generic vector field $X \in \mathscr{B}$ and $Y \neq a u_x$, $a \in \mathbb{R}$. Then Y is also generic and is determined by its leading terms.*

Proof. We only need to prove that symmetries are determined by their leading terms. Let $Y, Z \in \mathscr{B}$ be two symmetries of the generic vector field $X \in \mathscr{B}$ and have the same leading terms. Then from the identity

$$[X, Y - Z] = [X, Y] - [X, Z] = 0 \tag{3.7}$$

it follows that $W = Y - Z$ is also a symmetry of X. Since Y, Z have the same leading terms, the vector field W has the following expression

$$W = \sum_{d \geq 2} \epsilon^d W_d(u, u_1, \ldots, u_d). \tag{3.8}$$

Denote

$$X = \sum_{d \geq 1} \epsilon^d X_d(u, \ldots, u_d). \tag{3.9}$$

Then from (3.7) we know that the vector field $\epsilon^2 W_2$ is a symmetry of the vector field $\epsilon X_1(u, u_1)$. According to Lemma 3.3, symmetries of $\epsilon X_1(u, u_1)$ must be of degree 1, so we have $W_2 = 0$. Similarly, we can prove that all the W_m vanish and, consequently, $Y = Z$. The theorem is proved. □

Corollary 3.6. *Let $X \in \mathscr{B}$ be generic, and Y_1, Y_2 be two symmetries of X, then $[Y_1, Y_2] = 0$.*

Proof. By using the Jacobi identity, we know that $[Y_1, Y_2]$ is also a symmetry of X. But the leading term of $[Y_1, Y_2]$ vanishes, so from Theorem 3.5 it follows that $[Y_1, Y_2] = 0$. □

Remark 3.7. Let us denote $X \sim Y$ if $[X, Y] = 0$, then the above corollary shows that $\sim$ is an equivalence relation on the set of generic vector fields of $\mathscr{B}$.

Theorem 3.8. *For any generic $X \in \mathscr{B}$, there exists a Miura-type transformation g such that*

$$g(X) = \epsilon u u_1 + \epsilon^2 \left(f_{(2)} u_2 + f_{(1,1)} u_1^2\right) + \sum_{d \geq 3} \epsilon^d \left(\sum_{\lambda \in \mathscr{P}_1^d} f_\lambda(u) u_\lambda\right), \tag{3.10}$$

where $\mathscr{P}_1^d$ is the set of partitions of degree d whose non-zero entries are greater than 1.

Proof. Let $\epsilon f(u) u_1$ be the leading term of the vector field X. We can use a Miura-type transformation of the first kind $\tilde{u} = f(u)$ to transform X to the following form:

$$g(X) = \epsilon u u_1 + \sum_{d \geq 2} \epsilon^d \left(\sum_{\lambda \in \mathscr{P}^d} f_\lambda(u) u_\lambda\right). \tag{3.11}$$

Then, by using Lemma 3.4, we can find a series of Miura-type transformations of the second kind to eliminate the terms with factor u_1 step by step (except for the terms of degree 2). The theorem is proved. □

The expression (3.10) is called a reduced form of the vector field X. Note that such a reduced form may not be unique.

Definition 3.9. A generalized evolutionary PDE of the form (1.5) that corresponds to a generic vector field $X \in \mathscr{B}$ of the form (2.11) is called formally integrable, if any vector field of the form $Y = \epsilon h(u) u_1 \in \mathscr{B}$ can be extended to a symmetry

$$\tilde{Y} = \epsilon h(u) u_1 + \sum_{k \geq 2} \epsilon^k Y_k \in \mathscr{B} \tag{3.12}$$

of this equation.

By definition, a formally integrable equation of the form (1.5) must have infinitely many formal symmetries. Thus this definition is an adaption of the commonly used one of integrability from the point of view of symmetries in soliton theory, see for example [11,12,17–20] and references therein, to the present class of generalized evolutionary PDEs. In [11,12], it was conjectured by Fokas that the existence of a single time independent non-Lie point symmetry implies the existence of infinitely many for a scalar evolutionary PDE. We will reformulate this conjecture for our class of equations and their formal symmetries in Section 5. In [17,18,20], Mikhailov, Shabat and their collaborators formulated certain necessary conditions for the existence of a higher order symmetry for a system of evolutionary PDEs. These conditions are expressed in terms of the so called canonical conservation laws, which yield an effective algorithm of testing the existence of a higher order symmetry for certain class of evolutionary PDEs of lower order. In [19], Sanders and Wang gave a symbolic algorithm of checking the existence of symmetries for a certain class of scalar evolutionary PDE; such equations depend polynomially on the dependent variable and its x-derivatives, and satisfy certain homogeneity conditions. In Section 5, we will give an alternative way of checking the formal integrability of the generalized evolutionary PDEs; our approach is based on the quasi-triviality of such equations, which will be proved in the next section.

The following theorem relates the uniqueness of the reduced form of a vector field to the integrability of the corresponding PDE.

Theorem 3.10. *A generic vector field $X \in \mathscr{B}$ of the form (2.11) has a unique reduced form if and only if the corresponding Eq. (1.5) is formally integrable.*

Proof. Without loss of generality, we assume that X is already in a reduced form that is given by the right-hand side of (3.10).

Let us first assume that the reduced form of X is unique. Given a vector field

$$\epsilon Y_1 = \epsilon h(u) u_1 \in \mathscr{B},$$

consider the following vector field

$$g_1(X) := e^{\mathrm{ad}_{\epsilon Y_1}} X = \epsilon u u_1 + \epsilon^2 \left(f_{(2)} u_2 + f_{(1,1)} u_1^2 \right) + \cdots, \tag{3.13}$$

where g_1 corresponds to the Miura-type transformation

$$g_1 : u \mapsto e^{-\epsilon \hat{Y}_1} u. \tag{3.14}$$

From the proof of Theorem 3.8, we know that there exists a Miura-type transformation

$$g_2 : u \mapsto e^{\hat{Z}} u, \quad Z = \epsilon^2 Z_2 + \epsilon^3 Z_3 + \cdots \in \mathscr{B} \tag{3.15}$$

which transforms $g_1(X)$ to a reduced form. It follows from our assumption on the uniqueness of reduced form for the vector field X that

$$X = e^{-\mathrm{ad}_Z} g_1(X).$$

On the other hand, $e^{-\mathrm{ad}_Z} e^{\mathrm{ad}_{\epsilon Y_1}} X$ can be rewritten as $e^{\mathrm{ad}_Y} X$ with a vector field Y of the form

$$Y = \epsilon Y_1 + \epsilon^2 Y_2 + \cdots \in \mathscr{B}.$$

From the equality $X = e^{\mathrm{ad}_Y} X$, it then follows that the vector field X has a symmetry Y with the leading term ϵY_1. Thus, by our definition, the equation $\epsilon u_t = X$ is integrable.

Now let us assume that the equation $\epsilon u_t = X$ is integrable. If X has another reduced form $\tilde{X}$, then it must be related to X by a Miura-type transformation

$$\tilde{X} = e^{\mathrm{ad}_Y} X, \quad Y = \epsilon Y_1 + \epsilon^2 Y_2 + \cdots \in \mathscr{B}. \tag{3.16}$$

From the definition of integrability, we can find a symmetry Z of the vector field X that has the same leading term as that of the vector field Y, i.e.

$$Z = \epsilon Y_1 + \sum_{k \geq 2} \epsilon^k Z_k \in \mathscr{B}. \tag{3.17}$$

So we can express $\tilde{X}$ as

$$\tilde{X} = e^{\mathrm{ad}_Y} e^{-\mathrm{ad}_Z} X = e^{\mathrm{ad}_W} X. \tag{3.18}$$

Here, the vector field W has the form $W = \epsilon^2 W_2 + \cdots$. Since both X and $\tilde{X}$ are expressed in reduced forms, by using Lemma 3.2 we know that $W = 0$. Thus $\tilde{X} = X$, and we proved the uniqueness of reduced form of the vector field X. The theorem is proved. □

4. Quasi-triviality

We now proceed to consider the quasi-triviality of a generalized evolutionary PDE of the form (1.5) corresponding to a generic vector field X of the form (2.11). Define a map from the set of infinite series to itself

$$R : (y_1, y_2, \ldots) \mapsto (z_1, z_2, \ldots) \tag{4.1}$$

in the following recursive way

$$z_1 = \frac{1}{y_1}, \quad z_n = \frac{1}{y_1} \sum_{k \geq 1} y_{k+1} \frac{\partial z_{n-1}}{\partial y^k}, \quad n \geq 2. \tag{4.2}$$

When y_k are given by the kth order derivatives of a single variable smooth function A, then the above defined z_k are just the kth order derivatives of the inverse function of A. By using this observation, it is easy to see that R is an involution, i.e. R^2 is the identity map.

Lemma 4.1. *The equation* $[uu_1, f(u, u_1, \ldots)] = 0$ *for the unknown function* f *has the following general solution:*

$$f = \frac{1}{\phi_1} c(u, \phi_2, \phi_3, \ldots), \tag{4.3}$$

where c *is an arbitrary smooth function and* $(\phi_1, \phi_2, \ldots) = R(u_1, u_2, \ldots)$.

Proof. A solution of the equation $[uu_1, f(u, u_x, \ldots)] = 0$ corresponds to a flow $\frac{\partial u}{\partial s} = f$ that commutes with the flow $\frac{\partial u}{\partial t} = uu_x$. By performing a transformation

$$(x, t, s, u(x, t, s)) \mapsto (u, t, s, x(u, t, s)), \tag{4.4}$$

we can rewrite these two flows into the form

$$\frac{\partial x}{\partial s} = g(u, x_u, x_{uu}, \ldots), \quad \frac{\partial x}{\partial t} = -u. \tag{4.5}$$

Here, $g = -f(u, \frac{1}{x_u}, \frac{1}{x_u}\partial_u \frac{1}{x_u}, \ldots) \frac{\partial x}{\partial u}$. The condition of commutativity of the flows is given by

$$\frac{\partial g}{\partial t} = 0.$$

Due to the fact that $\frac{\partial}{\partial t} \frac{\partial^k x}{\partial u^k} = \frac{\partial^k}{\partial u^k} \frac{\partial x}{\partial t} = -\frac{\partial^k u}{\partial u^k} = -\delta_{k,1}$, we know that the general solution of the above equation has the form $g = -c(u, x_{uu}, x_{uuu}, \ldots)$ for certain smooth function c. So, f must have the form (4.3), and the lemma is proved. □

Proposition 4.2. *For any given smooth function* $F(u, u_1, \ldots)$, *the equation* $[uu_1, f] = F$ *for the unknown function* $f = f(u, u_1, \ldots)$ *has a particular solution*

$$f = -u_1 \left(\int^{\phi_1} \phi_1 F(u, \tilde{u}_1, \tilde{u}_2, \ldots) \mathrm{d}\phi_1 \right) \Bigg|_{(\phi_1, \phi_2, \ldots) = R(u_1, u_2, \ldots)}, \tag{4.6}$$

where $(\tilde{u}_1, \tilde{u}_2, \ldots) = R(\phi_1, \phi_2, \ldots)$.

Proof. Assume that we have a solution of the form $f = \frac{1}{\phi_1} c(u, \phi_1, \phi_2, \phi_3, \ldots)$. By using the above theorem, we know that the function c must satisfy the following equation:

$$\frac{\partial c}{\partial \phi_1} = -\phi_1 F(u, u_1, u_2, \ldots) = -\phi_1 F\left(u, \frac{1}{\phi_1}, -\frac{\phi_2}{\phi_1^3}, \ldots\right). \tag{4.7}$$

Here, in the second equality we used the substitution

$$(u_1, u_2, \ldots) = R(\phi_1, \phi_2, \ldots) = \left(\frac{1}{\phi_1}, -\frac{\phi_2}{\phi_1^3}, \ldots\right). \tag{4.8}$$

By integrating (4.7) with respect to ϕ_1 and making the substitution

$$(\phi_1, \phi_2, \ldots) = R(u_1, u_2, \ldots)$$

we finish the proof of the lemma. □

Now we are ready to prove the existence of a reducing transformation for the generalized evolutionary PDEs.

Theorem 4.3. *Given a generalized evolutionary PDE of the form*

$$u_t = u u_x + \epsilon\left(f_1(u) u_2 + f_2(u) u_1^2\right) + \sum_{k \geq 3} \epsilon^{k-1} S_k, \tag{4.9}$$

with $S_k \in \mathscr{R}$ being homogeneous differential polynomials of degree k, we have
(i) *there exists a unique transformation of the form*

$$u \mapsto v = e^{\hat{X}} u, \tag{4.10}$$

$$X = \sum_{k \geq 1} \epsilon^k u_1^{-L_k} \sum_{m=0}^{M_k} Y_{k,m}(u, u_1, \ldots, u_{N_k}) (\log u_1)^m \tag{4.11}$$

that reduces Eq. (4.9) to $v_t = v v_x$. Here, $Y_{k,m} \in \mathscr{R}$ are homogeneous differential polynomials of degree $L_k + k$, and L_k, M_k, N_k are integers that only depend on k.
(ii) *The transformation (4.10) and (4.11) also reduces any symmetry*

$$u_s = h(u) u_1 + \sum_{k \geq 2} \epsilon^{k-1} Q_k, \quad Q_k \in \mathscr{R}, \ \deg Q_k = k \tag{4.12}$$

of Eq. (4.9) to the form $v_s = h(v) v_x$.

Proof. Let the reducing transformation take the following form:

$$u \mapsto v = e^{\hat{X}} u = u + \hat{X}(u) + \frac{1}{2} \hat{X}(\hat{X}(u)) + \frac{1}{6} \hat{X}(\hat{X}(\hat{X}(u))) + \cdots \tag{4.13}$$

where $X = \epsilon X_1 + \epsilon^2 X_2 + \epsilon^3 X_3 + \epsilon^4 X_4 + \cdots$. It will eliminate all the perturbations, so we have

$$e^{\mathrm{ad}_X}(\epsilon S_1) = \epsilon S_1 + \epsilon^2 S_2 + \epsilon^3 S_3 + \epsilon^4 S_4 + \epsilon^5 S_5 + \cdots \tag{4.14}$$

where $S_1 = u u_1$, $S_2 = f_1(u) u_2 + f_2(u) u_1^2$. The coefficients of ϵ^k give us the following equations:

$$S_2 = [X_1, u u_1],$$

$$S_3 = [X_2, u u_1] + \frac{1}{2}[X_1, [X_1, u u_1]],$$

$$S_4 = [X_3, u u_1] + \frac{1}{2}[X_2, [X_1, u u_1]] + \frac{1}{2}[X_1, [X_2, u u_1]] + \frac{1}{6}[X_1, [X_1, [X_1, u u_1]]]$$

and so on. By using Proposition 4.2, we can solve these equations and obtain $X_1, X_2, X_3, X_4, \ldots$ recursively, and the homogeneity condition on $Y_{k,m}$ guarantees the uniqueness of the solution X_k, $k \geq 1$. Thus, we have proved the first part of the theorem.

To prove the second result of the theorem, let us note that, after the transformation (4.10) and (4.11), the Eq. (4.12) is transformed to

$$u_s = e^{-\mathrm{ad}_X}(h(u)u_1 + \sum_{k\geq 2}\epsilon^{k-1}Q_k) = h(u)u_x + \sum_{k\geq 2}\epsilon^{k-1}\tilde{Q}_k(u, u_1, \ldots, u_{m_k}) \tag{4.15}$$

where, for simplicity of notation, we keep to using the symbol u instead of v, and $\tilde{Q}_k$ are homogeneous polynomials of $\log u_1, u_1, \frac{1}{u_1}, u_2, \ldots, u_{m_k}$ of degree k (note that $\deg \log u_1 = 0$), where m_k is a positive integer depending on k. Since the transformation (4.10) and (4.11) reduces Eq. (4.9) to the form $u_t = uu_x$, we know that

$$\left[uu_x, h(u)u_x + \sum_{k\geq 2}\epsilon^{k-1}\tilde{Q}_k(u, u_1, \ldots, u_k)\right] = 0. \tag{4.16}$$

By using Lemma 4.1 and the fact that $\deg \tilde{Q}_k = k > 1$, we see that the functions $\tilde{Q}_k$ must vanish. The theorem is proved. □

Remark 4.4. We conjecture that the integers L_k, M_k, N_k have the following expressions:

$$L_k = 3k - 2, \quad M_k = 2\left[\frac{k-1}{2}\right] + \delta_{0,\left[\frac{k-1}{2}\right]}, \quad N_k = k + 1.$$

We hope that they can be proved by a careful analysis of the above procedure for constructing the reducing transformation.

By using Theorem 4.3, we are readily led to the following corollary:

Theorem 4.5. *Any generalized evolutionary PDE of the form (1.5) is quasi-trivial.*

As we have already noted in the introduction, the notion of quasi-Miura transformation that was introduced in [8] (see also [9]) requires that the functions $F_k(u, u_x, u_{xx}, \ldots)$ depend rationally on the variables $u_1, u_2, \ldots$. Here, we drop this rationality condition and still call (1.6) a quasi-Miura transformation. For Eq. (1.5), when $f_2 f' - f_1 f'' = 0$, the reducing transformation (1.6) depends rationally on the variables u_k, $k \geq 1$.

Example 4.6 (*KdV Equation*). The KdV Eq. (1.1) has the reducing transformation (1.2) and (1.4).

Example 4.7 (*Camassa–Holm Equation*). We consider another important equation in soliton theory, the Camassa–Holm equation [3,4,13]:

$$u_t - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0. \tag{4.17}$$

Let us perform the rescaling $t \mapsto -3\epsilon t,\ x \mapsto \epsilon x$, then the above equation can be put into the following form:

$$\begin{aligned} \epsilon u_t &= \left(1 - \epsilon\partial_x^2\right)^{-1}\left(\epsilon uu_1 - \epsilon^3\left(\frac{2}{3}u_1u_2 + \frac{1}{3}uu_3\right)\right) \\ &= \epsilon uu_1 + \epsilon^3\left(\frac{7u_1u_2}{3} + \frac{2uu_3}{3}\right) + \epsilon^5\left(\frac{23u_2u_3}{3} + \frac{11u_1u_4}{3} + \frac{2uu_5}{3}\right) + \cdots. \end{aligned} \tag{4.18}$$

This equation has the reducing transformation

$$\begin{aligned} v \mapsto u = v &+ \epsilon^2\left(\frac{7v_2}{6} - \frac{v{v_2}^2}{3{v_1}^2} + \frac{vv_3}{3v_1}\right) + \epsilon^4\left(\frac{6{v_2}^3}{5{v_1}^2} - \frac{202v{v_2}^4}{45{v_1}^4}\right. \\ &+ \frac{32v^2{v_2}^5}{9{v_1}^6} - \frac{181v_2v_3}{90v_1} + \frac{398v{v_2}^2v_3}{45{v_1}^3} - \frac{70v^2{v_2}^3v_3}{9{v_1}^5} - \frac{191v{v_3}^2}{90{v_1}^2} \\ &+ \frac{19v^2v_2{v_3}^2}{6{v_1}^4} + \frac{143v_4}{72} - \frac{133vv_2v_4}{45{v_1}^2} + \frac{34v^2{v_2}^2v_4}{15{v_1}^4} - \frac{73v^2v_3v_4}{90{v_1}^3} \\ &\left. + \frac{13vv_5}{18v_1} - \frac{41v^2v_2v_5}{90{v_1}^3} + \frac{v^2v_6}{18{v_1}^2}\right) + \cdots. \end{aligned}$$

Both the KdV equation and the Camassa–Holm equation have bihamiltonian structures; their quasi-triviality has been proved in [8] and [9]. The following two examples do not possess hamiltonian structures.

Example 4.8 (*Burgers Equation*). The Burgers equation

$$u_t = uu_x + \epsilon u_{xx} \tag{4.19}$$

has the reducing transformation

$$\begin{aligned} v \mapsto u = {} & v + \epsilon \frac{v_2}{v_1} + \epsilon^2 \left(\frac{2{v_2}^3}{{v_1}^4} - \frac{7v_2v_3}{3{v_1}^3} + \frac{v_4}{2{v_1}^2} \right) + \epsilon^3 \left(\frac{24{v_2}^5}{{v_1}^7} \right. \\ & \left. - \frac{46{v_2}^3v_3}{{v_1}^6} + \frac{16v_2{v_3}^2}{{v_1}^5} + \frac{34{v_2}^2v_4}{3{v_1}^5} - \frac{10v_3v_4}{3{v_1}^4} - \frac{11v_2v_5}{6{v_1}^4} + \frac{v_6}{6{v_1}^3} \right) \\ & + \epsilon^4 \left(\frac{568{v_2}^7}{{v_1}^{10}} - \frac{4544{v_2}^5v_3}{3{v_1}^9} + \frac{1086{v_2}^3{v_3}^2}{{v_1}^8} - \frac{179v_2{v_3}^3}{{v_1}^7} + \frac{1154{v_2}^4v_4}{3{v_1}^8} \right. \\ & - \frac{380{v_2}^2v_3v_4}{{v_1}^7} + \frac{221{v_3}^2v_4}{6{v_1}^6} + \frac{26v_2{v_4}^2}{{v_1}^6} - \frac{70{v_2}^3v_5}{{v_1}^7} + \frac{731v_2v_3v_5}{18{v_1}^6} \\ & \left. - \frac{101v_4v_5}{30{v_1}^5} + \frac{83{v_2}^2v_6}{9{v_1}^6} - \frac{13v_3v_6}{6{v_1}^5} - \frac{5v_2v_7}{6{v_1}^5} + \frac{v_8}{24{v_1}^4} \right) + \cdots . \end{aligned}$$

Example 4.9. Our last example is a class of equations parameterized by a smooth function f:

$$u_{t_f} = u_1 f(u + \epsilon u_1) = \sum_{n \geq 0} \epsilon^n \frac{f^{(n)}(u)}{n!} {u_1}^{n+1}. \tag{4.20}$$

It is easy to verify that ∂_{t_f} and ∂_{t_g} commute for arbitrary smooth functions f and g. The reducing transformation of this equation has an explicit form

$$v \mapsto u = e^{\epsilon \hat{X}}(v) = v + \epsilon \hat{X}(v) + \frac{\epsilon^2}{2} \hat{X}(\hat{X}(v)) + \frac{\epsilon^3}{6} \hat{X}(\hat{X}(\hat{X}(v))) + \cdots , \tag{4.21}$$

where $X = v_1 \log(v_1)$. This equation is quite different to those of the above three examples, because its reducing transformation contains $\log(v_1)$, while that of the KdV, the Camassa–Holm and the Burgers equations are rational in the jet variables $v_1, v_2, \ldots$.

5. Testing of integrability

Given a generalized evolutionary PDE of the form (1.5), let

$$g : u \mapsto v$$

be the quasi-Miura transformation (1.6) that reduces it to the form (1.7). For any smooth function $h(v)$, Eq. (1.7) has a symmetry

$$v_s = h(v) v_x. \tag{5.1}$$

Rewriting this flow in the u coordinates by using the quasi-Miura transformation g^{-1}, we have

$$\epsilon u_s = \epsilon h(u) u_x + \sum_{k \geq 2} \epsilon^k W_k(u, u_1, \ldots, m_k), \tag{5.2}$$

where, in general, W_k are not polynomials of the variables $u_1, u_2, \ldots$.

In order to compute these W_k in a more direct way, we first perform a change of the dependent variable

$$u \mapsto f(u) \tag{5.3}$$

to transform the Eq. (1.5) into the form

$$\epsilon u_t = \tilde{X}, \quad \tilde{X} = \epsilon u u_x + \sum_{k\geq 2} \epsilon^k \tilde{X}_k(u, \ldots, u_k). \tag{5.4}$$

Consider the equation $[\tilde{X}, Y] = 0$ for the unknown vector field

$$Y = \epsilon h \circ f^{-1}(u) u_x + \sum_{k\geq 2} \epsilon^k Y_k(u, \ldots, u_{n_k}).$$

By using Proposition 4.2, we can solve this equation recursively to obtain Y. Under a certain homogeneity condition that is similar to the one given in Theorem 4.3, the solution Y is unique. By performing the transformation $u \mapsto f^{-1}(u)$, the vector field Y is transformed to the form

$$\epsilon h(u) u_x + \sum_{k\geq 2} \epsilon^k \tilde{Y}_k(u, \ldots, u_{n_k}).$$

Then we have $W_k = \tilde{Y}_k$.

Although the above flow (5.2) commutes with the flow given by Eq. (1.5), in general it does not meet the requirement of being a formal symmetry of (1.5) according to our Definition 3.1. This is because, in general, the right-hand side of Eq. (5.2) does not belong to $\mathscr{B}$, i.e. the functions $W_k(u, u_1, \ldots, u_{m_k})$ are not polynomials of the variables $u_1, u_2, \ldots, u_{m_k}$.

Now let us come back to our Definition 3.9 of formal integrability for a generalized evolutionary PDE of the form (1.5). By using the second result of Theorem 4.3 and the quasi-triviality of the Eq. (1.5), we know that if a vector field $\epsilon h(u) u_x$ can be extended to a formal symmetry of the Eq. (1.5), then this formal symmetry must coincide with (5.2). Thus we have the following:

Criterion of formal integrability. *A generalized evolutionary PDE of the form (1.5) is formally integrable iff, for any smooth function $h(u)$, the functions W_k that appear in the right-hand side of (5.2) are homogeneous differential polynomials of $u_1, u_2, \ldots, u_k$ of degree k.*

Let us consider the formal integrability of the four equations considered in the last section. The formal integrability of the equation given in the fourth example is trivial, because we can write down all its symmetries explicitly. For the other three equations, we have the following proposition:

Proposition 5.1. *The KdV Eq. (1.1), the Camassa–Holm equation (4.17) and the Burgers equation (4.19) are formally integrable.*

Proof. According to Proposition 4.2 and the above construction, we know that the flow (5.2) for the KdV equation must take the following form:

$$u_s = h(u) u_x + \sum_{k\geq 1} \epsilon^{2k} \frac{1}{u_x^{m_k}} \sum_{\lambda \in \mathscr{P}^{m_k+2k+1}} C_{k,\lambda} h^{(D_{k,\lambda})}(u) u_\lambda, \tag{5.5}$$

where m_k are integers, $C_{k,\lambda}$ are rational numbers, and $D_{k,\lambda}$ are positive integers. To prove the theorem, we only need to show that, for each monomial

$$u_x^{-m_k} C_{k,\lambda} h^{(D_{k,\lambda})}(u) u_\lambda \tag{5.6}$$

that appears in the above expression, either $C_{k,\lambda} = 0$ or it can be reduced to the form

$$C_{k,\lambda} h^{(D_{k,\lambda})}(u) u_{\lambda'}, \quad \lambda' \in \mathscr{P}.$$

To this end, let us take $h(u) = u^{D_{k,\lambda}}$. From the classical theory of the KdV equation, we know that the flow (5.5) belongs to the KdV hierarchy and so its right-hand side is a truncated differential polynomial. So the monomial (5.6) has the required property and we have proved the formal integrability of the KdV equation.

In a similar way, we can prove the formal integrability of the Camassa–Holm equation and the Burgers equation. The only technical point we should note is that the analogue of (5.5) for the Camassa–Holm equation needs to be modified slightly. We omit the details here. The proposition is proved. □

The following two examples illustrate a procedure for identifying the formally integrable equations among those that possess certain particular forms.

Example 5.2. Consider the formally integrable equations among those of the form

$$u_t = uu_x + \epsilon(f_1(u)u_{xx} + f_2(u)u_x^2), \quad f_1(u) \neq 0. \tag{5.7}$$

By using the criterion of formal integrability, we easily obtain

$$f_1(u) = au + b, \quad f_2(u) = -\frac{1}{2}a, \tag{5.8}$$

where a, b are arbitrary constants. When $a = 0, b = 1$, the above equation is just the well-known Burgers equation (4.19). If $a \neq 0$, we may assume, without loss of generality, that $a = 1$. The Galilean transformation $x \mapsto x - bt,\ t \mapsto t,\ u \mapsto u - b$ converts (5.7) to the form

$$u_t = uu_x + \epsilon\left(uu_{xx} - \frac{1}{2}u_x^2\right). \tag{5.9}$$

Letting $u = w^2$, Eq. (5.9) becomes

$$w_t = w^2\left(w_x + \epsilon w_{xx}\right). \tag{5.10}$$

This equation is also linearizable under the change of the independent variables and is called C-integrable; see [2] and equation (3.37) therein.

Eq. (5.10) admits a recursion operator

$$R = \epsilon w \partial_x + w + w^2\left(w_x + \epsilon w_{xx}\right)\partial_x^{-1}\frac{1}{w^2}. \tag{5.11}$$

One can prove that R is a hereditary strong symmetry [13]. So, we obtain a hierarchy of symmetries of Eq. (5.10) which can be expressed as

$$w_{t_0} = -w_x, \quad w_{t_1} = 0, \quad w_{t_{2+k}} = R^k\left(w^2\left(w_x + \epsilon w_{xx}\right)\right) = (k+1)w^{k+2}w_x + \cdots, \quad k \geq 0. \tag{5.12}$$

To see that all the right-hand sides of these symmetries are differential polynomials of the variables $w_x, w_{xx}, \ldots$, we introduce a generating function

$$F = \sum_{k=0}^{\infty}\frac{w_{t_{k+2}}}{\lambda^{k+2}} = \frac{w_{t_2}}{\lambda^2} + \frac{w_{t_3}}{\lambda^3} + \cdots.$$

This function satisfies the following equation:

$$RF = \lambda\left(F - \frac{w_{t_2}}{\lambda^2}\right). \tag{5.13}$$

Letting $w_{t_k} = w^2\partial_x h_k$, $H = \sum_{k\geq 2}\frac{h_k}{\lambda^k}$, then Eq. (5.13) becomes a linear ordinary differential equation (ODE) of H. Its solution is

$$H = \frac{1}{w}\sum_{k=0}^{\infty}\epsilon^k\left(A\partial_x\right)^k\left(A\frac{w+\epsilon w_x}{\lambda}\right), \quad \text{where } A = \sum_{l=1}^{\infty}\frac{w^l}{\lambda^l}.$$

This implies that all the right-hand sides of the flows given in Eq. (5.12) are homogeneous differential polynomials. By using a similar argument as given in the proof of Proposition 5.1, one can prove that Eq. (5.10) is formally integrable.

The integrable hierarchy (5.12) has the conservation law

$$\left(-\frac{1}{w}\right)_{t_k} = \partial_x h_k. \tag{5.14}$$

This defines a reciprocal transformation

$$\mathrm{d}y = \frac{1}{w}\mathrm{d}x - \sum_{k\geq 2} h_k \mathrm{d}t_k \tag{5.15}$$

which transforms the whole hierarchy to the Burgers hierarchy (up to a rescaling). In the next example, we will give more detailed explanations on the reciprocal transformation.

In the above example, the conditions (5.8) can in fact be derived by requiring the existence of a function $h(u)$ with $h''(u) \neq 0$ such that the resulting functions W_k are differential polynomials. A similar situation also occurs in other examples that we computed, such as the one that will be presented below. Based on these examples, we reformulate the conjecture of Fokas [11,12] on the existence of infinitely many symmetries of a scalar evolutionary PDE for the class of equations considered in this paper as follows:

Conjecture 5.3. *A generalized evolutionary PDE of the form* (1.5) *is formally integrable iff there exists a smooth function* $h(u)$ *satisfying*

$$h'' - \frac{h'}{f'} f'' \neq 0, \tag{5.16}$$

such that the flow (5.2) *that is obtained from Eq.* (5.1) *by the reducing transformation of* (1.5) *gives a symmetry of* (1.5)*, i.e. the functions* W_k *are homogeneous differential polynomials of degree* k.

Example 5.4. Consider the equation of the form

$$u_t = u u_x + \epsilon^2 \left(g_1(u) u_{xxx} + g_2(u) u_x u_{xx} + g_3(u) u_x^3 \right), \quad g_1(u) \neq 0. \tag{5.17}$$

Lemma 5.5. *If Eq.* (5.17) *is formally integrable, then the functions* g_1, g_2, g_3 *must satisfy the equations*

$$9g_1^2 g_2'' - 6g_1 g_2 g_1'' - 9g_1 g_1' g_2' - 18 g_1 g_2 g_2' + 8 g_2 g_1'^2 + 12 g_2^2 g_1' + 4 g_2^3 = 0, \tag{5.18}$$

$$12 g_1^2 g_1''' - 8 g_1 g_1' g_1'' - 12 g_1 g_2 g_1'' - 3 g_1 g_1' g_2' + 4 g_1'^3 + 6 g_2 g_1'^2 + 2 g_2^2 g_1' = 0 \tag{5.19}$$

and the relation

$$g_3 = \frac{1}{72 g_1} \left(6 g_2^2 - 30 g_2 g_1' - 4 g_1'^2 + 27 g_1 g_2' + 12 g_1 g_1'' \right). \tag{5.20}$$

Proof. By using the criterion of formal integrability, we arrive at the result of the lemma from the polynomiality property of the first few W_k. The lemma is proved. □

Lemma 5.6. *If the conditions* (5.19) *and* (5.20) *hold true and* $g_1(u)$ *is not a constant, then there exists a reciprocal transformation*

$$\mathrm{d}y = f(u)\mathrm{d}x + \rho(u, u_x, u_{xx})\mathrm{d}t, \quad \mathrm{d}s = \mathrm{d}t, \tag{5.21}$$

such that Eq. (5.17) *is transformed to an equation of the following form:*

$$u_s = \tilde{f}(u) u_y + \epsilon^2 \left(u_{yyy} + \tilde{g}_2(u) u_{yy} u_y + \tilde{g}_3(u) u_y^3 \right). \tag{5.22}$$

Proof. To define the reciprocal transformation (5.21), $f(u)$ must be the density of a conservation law, i.e. there exists a function $\rho(u, u_x, u_{xx})$ such that

$$(f(u))_t = (\rho(u, u_x, u_{xx}))_x .$$

By using Eq. (5.17), it is easy to see that $f(u)$ must satisfy the following equation:

$$g_1 f''' + 2 g_1' f'' + g_1'' f' - g_2 f'' - g_2' f' + 2 g_3 f' = 0, \tag{5.23}$$

and the flux ρ is given by

$$\rho = uf - \tilde{f} + \epsilon^2 \left(g_1 f' u_{xx} + \frac{1}{2} \left(g_2 f' - g_1 f'' - g_1' f' \right) u_x^2 \right), \tag{5.24}$$

where $\tilde{f}(u) = \int f(u)\mathrm{d}u$.

Suppose that we have a function $f(u)$ satisfying (5.23) and a function ρ given by (5.24), then the reciprocal transformation (5.21) is well defined and it converts Eq. (5.17) to the following equation:

$$u_s = \tilde{f}(u)u_y + \epsilon^2 \left(g_1(u) f(u)^3 u_{yyy} + \tilde{g}_2(u) u_{yy} u_y + \tilde{g}_3(u) u_y^3 \right). \tag{5.25}$$

To complete the proof, we only need to show that $f = g_1^{-\frac{1}{3}}$ is a solution of Eq. (5.23). In fact, after the substitutions of $f = g_1^{-\frac{1}{3}}$ and (5.20) into Eq. (5.23), we obtain an equation that is equivalent to (5.19). The lemma is proved. □

Now let us perform a Miura-type transformation of the first kind

$$u \mapsto \tilde{u} = \tilde{f}(u)$$

to Eq. (5.22). We obtain an equation of the following form:

$$\tilde{u}_s = \tilde{u}\tilde{u}_y + \epsilon^2 \left(\tilde{u}_{yyy} + \bar{g}_2(\tilde{u}) \tilde{u}_{yy} \tilde{u}_y + \bar{g}_3(\tilde{u}) \tilde{u}_y^3 \right). \tag{5.26}$$

Since the above reciprocal and Miura-type transformations keep the formal integrability property, the functions $\bar{g}_1 = 1, \bar{g}_2, \bar{g}_3$ also satisfy the conditions (5.18)–(5.20), i.e.

$$\bar{g}_2'' - 2\bar{g}_2 \bar{g}_2' + \frac{4}{9} \bar{g}_2^3 = 0, \tag{5.27}$$

$$\bar{g}_3 = \frac{1}{24} \left(2\bar{g}_2^2 + 9\bar{g}_2' \right). \tag{5.28}$$

Lemma 5.7. *Suppose the functions $\bar{g}_2(\tilde{u}), \bar{g}_3(\tilde{u})$ satisfy the conditions (5.27), (5.28), then there exists a Miura-type transformation*

$$u \mapsto \bar{u} = \tilde{u} + \epsilon f_0(\tilde{u})\tilde{u}_y + \epsilon^2 \left(f_1(\tilde{u}) \tilde{u}_{yy} + f_2(\tilde{u}) \tilde{u}_y^2 \right) \tag{5.29}$$

that transforms Eq. (5.26) to the KdV equation

$$\bar{u}_s = \bar{u}\bar{u}_y + \epsilon^2 \bar{u}_{yyy}.$$

Proof. The ODE (5.27) has the following general solution:

$$\bar{g}_2(\tilde{u}) = -\frac{3 f'(\tilde{u})}{2 f(\tilde{u})}, \quad f(\tilde{u}) = a_0 + a_1 \tilde{u} + a_2 \tilde{u}^2,$$

where a_0, a_1, a_2 are arbitrary constants. Let us take

$$f_0 = \frac{\sqrt{\frac{3}{2}} \sqrt{4a_0 a_2 - a_1^2}}{\sqrt{f} \sqrt{f' + 2\sqrt{a_2}\sqrt{f}}}, \quad f_1 = \frac{3\sqrt{a_2}}{2\sqrt{f}}, \quad f_2 = f_1' - \frac{1}{6} f_1^2.$$

The lemma is proved after a straightforward verification. □

The above three Lemmas show that Eq. (5.17) is formally integrable if the functions g_1, g_2, g_3 satisfy (5.18)–(5.20), and that modulo reciprocal and Miura-type transformations there is only one formally integrable equation, namely, the KdV equation. So our formal integrability in fact leads to integrable equations in the common sense.

To see some concrete formally integrable equations of the form (5.17) that are in different guises of the KdV equation, let us further impose the additional conditions on the functions g_1, g_2, g_3 by requiring that the right-hand side of (5.17) is graded homogeneous with respect to the following grading:

$$\deg u = 1, \quad \deg \partial_x^m u = 1 - m, \quad \deg \epsilon = 1 + \delta, \tag{5.30}$$

where δ is a certain constant. Since $g_1(u) \neq 0$, we can take

$$g_1(u) = u^{1-2\delta}, \quad g_2(u) = \alpha u^{-2\delta}, \quad \alpha \text{ is constant.} \tag{5.31}$$

Then Eqs. (5.18), (5.19) have the following solutions (α, δ):

$$\left(0, \frac{1}{2}\right), \ \left(-\frac{3}{2}, \frac{1}{2}\right), \ \left(-3, \frac{1}{2}\right), \ (-2, 0), \ (-1, 0), \ \left(0, -\frac{1}{4}\right),$$
$$\left(-\frac{3}{4}, -\frac{1}{4}\right), \ \left(0, -\frac{1}{2}\right), \ (-6, 2).$$

The associated equations are given by

$$\left(0, \frac{1}{2}\right) : u_t = uu_x + \epsilon^2 u_{xxx}, \tag{5.32}$$

$$\left(-\frac{3}{2}, \frac{1}{2}\right) : w_t = w^2 w_x + \epsilon^2 w_{xxx}, \quad w^2 = u, \tag{5.33}$$

$$\left(-3, \frac{1}{2}\right) : u_t = uu_x + \epsilon^2 \left(u_{xxx} - \frac{3}{u} u_x u_{xx} + \frac{15}{8u^2} u_x^3\right), \tag{5.34}$$

$$(-2, 0) : w_t = w^3 w_x + \epsilon^2 w^3 w_{xxx}, \quad w^3 = u, \tag{5.35}$$

$$(-1, 0) : w_t = w^3 w_x + \epsilon^2 \left(3w^2 w_x w_{xx} + w^3 w_{xxx}\right), \quad w^3 = u, \tag{5.36}$$

$$\left(0, -\frac{1}{4}\right) : w_t = w^2 w_x + \epsilon^2 \left(3w^2 w_x w_{xx} + w^3 w_{xxx}\right), \quad w^2 = u, \tag{5.37}$$

$$\left(-\frac{3}{4}, -\frac{1}{4}\right) : w_t = w^2 w_x + \epsilon^2 \left(\frac{3}{2} w^2 w_x w_{xx} + w^3 w_{xxx}\right), \quad w^2 = u, \tag{5.38}$$

$$\left(0, -\frac{1}{2}\right) : u_t = uu_x + \epsilon^2 \left(u^2 u_{xxx} + \frac{1}{9} u_x^3\right), \tag{5.39}$$

$$(-6, 2) : w_t = \frac{w_x}{w} + \epsilon^2 w^3 w_{xxx}, \quad \frac{1}{w} = u. \tag{5.40}$$

The first two Eqs. (5.32) and (5.33) are the KdV and mKdV equations, respectively. The third one (5.34) is equivalent to the KdV Eq. (5.32) by the following Miura-type transformation:

$$u \mapsto \tilde{u} = u + \epsilon^2 \left(\frac{3u_{xx}}{2u} - \frac{15u_x^2}{8u^2}\right), \tag{5.41}$$

where u satisfies (5.34) and $\tilde{u}$ satisfies (5.32).

Eqs. (5.32)–(5.34) have the following pairs of conserved quantities, respectively:

$$\int u \mathrm{d}x, \ \int u^2 \mathrm{d}x; \ \int w \mathrm{d}x, \ \int w^2 \mathrm{d}x; \ \int \sqrt{u} \mathrm{d}x, \ \int \frac{1}{\sqrt{u}} \mathrm{d}x.$$

By using these conserved quantities, we can define six reciprocal transformations that relate the remaining six Eqs. (5.35)–(5.40) to the first three Eqs. (5.32)–(5.34). Let us consider in detail the reciprocal transformation that is defined by the second conserved quantity $\int u^2(x, t) \mathrm{d}x$ of Eq. (5.32). The conservation law is given by

$$\left(u^2\right)_t = \rho_x, \quad \text{where } \rho = \frac{2}{3} u^3 + \epsilon^2 \left(2uu_{xx} - u_x^2\right). \tag{5.42}$$

Thus we can define the following reciprocal transformation:

$$\mathrm{d}y = u^2 \mathrm{d}x + \rho \mathrm{d}t, \quad \mathrm{d}s = \mathrm{d}t. \tag{5.43}$$

Its inverse is given by

$$\mathrm{d}x = \frac{1}{u^2}\mathrm{d}y - \left(\frac{2}{3}u + \epsilon^2\left(\frac{2u_{xx}}{u} - \frac{u_x^2}{u^2}\right)\right)\mathrm{d}s, \quad \mathrm{d}t = \mathrm{d}s. \tag{5.44}$$

From definition (5.43), we obtain $\partial_x = u^2\partial_y$, so

$$u_x = u^2 u_y, \quad u_{xx} = u^2\left(u^2 u_{yy} + 2u u_y^2\right). \tag{5.45}$$

Now the reciprocal transformation (5.44) can be rewritten as

$$\mathrm{d}x = \frac{1}{u^2}\mathrm{d}y - \left(\frac{2}{3}u + \epsilon^2\left(2u^3 u_{yy} + 3u^2 u_y^2\right)\right)\mathrm{d}s, \quad \mathrm{d}t = \mathrm{d}s. \tag{5.46}$$

So, in terms of the new independent variables y, s, the function u satisfies the following equation:

$$u_s = \frac{u^3}{3}u_y + \epsilon^2\left(u^6 u_{yyy} + 6u^5 u_y u_{yy} + 3u^4 u_y^3\right). \tag{5.47}$$

By performing the rescaling $s \mapsto \tilde{s} = s/3, \epsilon \mapsto \tilde{\epsilon} = \sqrt{3}\epsilon$ and the Miura-type transformation of first kind $u \mapsto \tilde{u} = u^3$, the above equation is transformed to

$$\tilde{u}_{\tilde{s}} = \tilde{u}\tilde{u}_y + \tilde{\epsilon}^2\left(\tilde{u}^2\tilde{u}_{yyy} + \frac{1}{9}\tilde{u}_y^3\right), \tag{5.48}$$

which coincides with Eq. (5.39).

For the other cases, we only point out the following facts:

1. Eq. (5.32) is transformed to Eq. (5.37) by the reciprocal transformation defined by the conserved quantity $\int u\mathrm{d}x$.
2. Eq. (5.33) is transformed to Eqs. (5.36) and (5.38), respectively, by the reciprocal transformations defined by the conserved quantities $\int w\mathrm{d}x$ and $\int w^2\mathrm{d}x$.
3. Eq. (5.34) is transformed to Eqs. (5.35) and (5.40), respectively, by the reciprocal transformations defined by the conserved quantities $\int \sqrt{u}\mathrm{d}x$ and $\int 1/\sqrt{u}\mathrm{d}x$.

Note that Eq. (5.36) and (5.37) are two cases of the S-integrable equation (3.55) given in [2].

The analysis of the above two examples shows that the formal integrability condition is rather rigid. Modulo the Miura-type transformations and reciprocal transformations, the Burgers equation and the KdV equation are the only integrable equations among the two classes of equations of the form (5.7) and (5.17). The following conjecture generalizes the results of the first example:

Conjecture 5.8. *All the formally integrable equations written in the reduced form*

$$u_t = uu_x + \epsilon(f_2(u)u_2 + f_{11}(u)u_1^2) + \epsilon^2 f_3(u)u_3 + \epsilon^3(f_4(u)u_4 + f_{22}(u)u_2^2) + \cdots \tag{5.49}$$

with $f_2(u) \neq 0$ are parameterized by the functions $f_2(u)$ and $f_{11}(u)$. In other words, the formal integrability condition for Eq. (5.49) with $f_2(u) \neq 0$ uniquely determines all the coefficient functions $f_\lambda(u)$ with $|\lambda| > 2$ through the functions $f_2(u)$ and $f_{11}(u)$, for example,

$$\begin{aligned} f_3 &= \frac{4}{3}f_2 f_{11} + \frac{2}{3}f_2 f_2', \\ f_4 &= \frac{5}{3}f_2^2 f_{11}' + \frac{1}{3}f_2^2 f_2'' + 2f_2 f_{11}^2 + \frac{1}{3}f_2(f_2')^2 + \frac{5}{3}f_{11}f_2 f_2', \\ f_{22} &= -\frac{10}{3}f_{11}^3 - f_{11}^2 f_2' + \frac{1}{3}f_{11}(f_2')^2 - \frac{29}{3}f_2 f_{11} f_{11}' - \frac{10}{3}f_2 f_{11} f_2'' \\ &\quad - 4f_2 f_2' f_{11}' - f_2 f_2' f_2'' - 3f_2^2 f_{11}'' - f_2^2 f_2''', \ldots. \end{aligned}$$

Moreover, modulo a certain reciprocal transformation, a formal integrable equation of the above form is equivalent, under a Miura-type transformation, to a formal symmetry of the Burgers equation (4.19).

The parameter space for the class of integrable equations of the form (5.49) with $f_2(u) = 0$ is as yet unknown. It seems that there exist infinitely many function parameters in this space. We hope that the additional condition of possessing a hamiltonian structure will restrict this parameter space to a manageable one. We will consider this problem in subsequent publications.

6. Conclusion

We have proved the quasi-triviality of a generalized evolutionary PDE of the form (1.5) and proposed a criterion for its integrability. Due to Proposition 4.2, the proof of Theorem 4.3 gives a constructive algorithm to obtain an explicit expression of the quasi-Miura transformation that reduces Eq. (1.5) to its leading term Eq. (1.7). Also, due to Proposition 4.2, the explicit expression of the flows of the form (5.2) that commute with (1.5) can be obtained in a direct way, as explained in the beginning of Section 5, thus the algorithm of testing the formal integrability of an equation of the form (1.5) can be encoded into a simple computer program.

Now, it is natural to ask the question whether the quasi-triviality property can be generalized to systems of generalized evolutionary PDEs of the form

$$u_t^i = \sum_{j=1}^n \lambda_j^i(\mathbf{u}) u_x^j + \sum_{k\geq 1} \epsilon^k Q_k^i(\mathbf{u}, \mathbf{u}_x, \ldots, \mathbf{u}^{(k+1)}), \quad i = 1, \ldots, n \geq 2, \tag{6.1}$$

where $\mathbf{u} = (u^1, \ldots, u^n)$, λ_j^i are smooth functions of $\mathbf{u}$ and Q_k^i are homogeneous polynomials of $\partial_x^m u^l$, $l = 1, \ldots, n, m = 1, \ldots, k+1$ with degree $k+1$, where we define $\deg \partial_x^m u^l = m$. It is proved in [9] that, if the above system has a semisimple bihamiltonian structure, then it is quasi-trivial, i.e. there exists a quasi-Miura transformation of the form

$$u^i \mapsto v^i = u^i + \sum_{k\geq 1} \epsilon^k F_k^i(\mathbf{u}, \mathbf{u}_x, \ldots), \quad i = 1, \ldots, n \tag{6.2}$$

that reduces the above system to a system given by its leading terms

$$v_t^i = \sum_{j=1}^n \lambda_j^i(\mathbf{v}) v_x^j. \tag{6.3}$$

We expect that the condition of possessing a semisimple bihamiltonian structure could be replaced by a much weaker one in order to ensure the quasi-triviality of the systems of the above form.

The definition of formal integrability for equations of the form (1.5) can be directly generalized to systems of equations of the form (6.1). We plan to study the problem of its quasi-triviality and integrability in subsequent publications.

Acknowledgments

The authors are grateful to Boris Dubrovin for encouragement and helpful discussions. The research of Y.Z. was partially supported by the Chinese National Science Fund for Distinguished Young Scholars grant No.10025101 and the Special Funds of Chinese Major Basic Research Project "Nonlinear Sciences".

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